A Collineation Group Isomorphic with the Group of the Double Tangents of the Plane Quartic.

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Introduction.

The group of the double tangents of a plane quartic is isomorphic with one of a series of groups arising in connection with the theta functions. This one is associated with the division into half-periods for $p=3$. Its immediate predecessor associated analytically with the division into half-periods for $p=2$ is the group of order $16 \cdot 720$ associated geometrically with the Kummer * surface. A similar one † is determined by the division into thirds of periods for theta functions for $p=2$, and is associated geometrically with the lines of a cubic surface. In all these cases isomorphic collineation groups have been discovered and discussed in considerable detail, but no collineation group isomorphic with the group of the double tangents has been discussed. It is the purpose of this paper to derive such a group. The group being connected with the quartic curve, by proper mapping methods a collineation group is obtained in which the variables are irrational invariants of the quartic itself. The equation of the quartic and its double tangents are obtained in a form whose symmetry and simplicity leave nothing to be desired. A complete system for the collineation group and associated canonical forms of the quartic are obtained. The collineation group appears in seven variables. That this is the smallest number of variables in which this group can be represented as a collineation group is evident from a theorem of Wiman in Weber, "Lehrbuch der Algebra," Vol. II, p. 376. The results obtained are applicable to the solution of the equation of the double tangents of the quartic, and should also be valuable for discussing certain invariants of the quartic and configurations of the double tangents. The quartic appears with an isolated flex and may throw some light on the hitherto unsolved problem of the flexes.

* An account of this group in relation to the Kummer surface is to be found in Hudson's "Kummer's Quartic Surface," which appeared in 1905.

† This group was discussed by Burkhardt, who gave an historical account of the matter up to the time of his papers (about 1890). They appeared in the *Math. Annalen*, Vols. XXXV, XXXVI, XXXVIII.

I.

The Cremona Group $G_{7,2}$ of P_7^2 in S_6.

Two sets of seven points in a plane, P_7^2 and Q_7^2, ordered with respect to each other, are *congruent under the Cremona transformation* C_m with ρ F-points if ρ of the pairs p_i, q_i $(i=1, 2, \ldots, 7)$ are corresponding F-points of C_m, and if the remaining $7-\rho \geqq 0$ of the pairs p_i, q_i are pairs of ordinary corresponding points under C_m. The number of projectively distinct sets congruent to P_7^2 is the number of types of Cremona transformations. To determine this number the following theorem* is necessary:

The general Cremona transformation C_m $(m>2)$ with ρ F-points is projectively determined when there are given the order m, the ρ F-points, their multiplicities subject to the conditions $\sum_1^\rho r_i^2 = m^2-1$, $\sum_1^\rho r_i = 3(m-1)$, and the positions of four corresponding F-points.

The possible transformations to be considered in connection with P_7^2 are given by the following table where α_j is the number of F-points of multiplicity j:

	C_2	C_3	C_4	D_4	C_5	D_5	D_6	D_7	D_8
α_1	3	4	3	6	0	3	1	0	0
α_2		1	3	0	6	3	4	3	0
α_3				1		1	2	4	7

C is used to indicate a transformation with six or fewer F-points, D one with seven F-points. Using in addition the collineation C_1 we find the number of transformations C_1, C_2, C_3, C_4, C_5, D_4, D_5, D_6, D_7, D_8 to be respectively $\binom{7}{0}$, $\binom{7}{3}$, $\binom{7}{4}\binom{3}{1}$, $\binom{7}{3}\binom{4}{3}$, $\binom{7}{6}$, $\binom{7}{6}$, $\binom{7}{3}\binom{4}{3}$, $\binom{7}{1}\binom{6}{4}$, $\binom{7}{3}$, $\binom{7}{0}$ or 2.288. But since P_7^2 and Q_7^2 congruent under D_8 are projective, there are only 288 projectively distinct types of congruence.

The sets P_7^2 and Q_7^2 are mapped upon the points of a space S_6 by taking them in the canonical form:

$$P_7^2\colon (1,0,0), \quad (0,1,0), \quad (0,0,1), \quad (1,1,1), \quad (x_i, y_i, u), \quad (i=1,2,3),$$
$$Q_7^2\colon (1,0,0), \quad (0,1,0), \quad (0,0,1), \quad (1,1,1), \quad (x_j', y_j', u), \quad (j=1,2,3),$$

and regarding $x_1, x_2, x_3, y_1, y_2, y_3, u$ as the coordinates of a point in S_6. Then, if two sets of points are congruent under a Cremona transformation in S_2, their maps in S_6 are corresponding points under a Cremona transformation

* Coble, "Point Sets and Allied Cremona Groups," Part II, *Transactions of the American Mathematical Society*, Vol. XVII, p. 348.

in S_6. The general Cremona transformation in S_2 can be expressed as a product of quadratic factors. The effect in S_6 corresponding to a quadratic transformation in S_2 is that of an involutory Cremona transformation. Moreover, any Cremona transformation of the kind considered is a product of transformations corresponding to quadratic transformations in S_2. Any quadratic transformation in S_2 with F-points at points of P_7^2 can be obtained from a single one by permutation of the points. Hence $G_{7,2}$, the Cremona group of P_7^2 in S_6, can be generated by the symmetric group of order $7!$ and a single transformation in S_6 corresponding to a quadratic transformation in S_2. The number of operations in $G_{7,2}$ is clearly the same as the number of types of congruence of sets P_7^2 if further it is required that P_7^2 be ordered. $G_{7,2}$ is thus seen to be of order $7!288$.

II.

Point Sets on a Cuspidal Cubic.

The cuspidal cubic curve $C_1 \equiv x_2^3 - x_1 x_3^2 = 0$ is given parametrically by $x_1 = t^3$; $x_2 = t$; $x_3 = 1$, the parameter of the cusp being $t = \infty$ and that of the flex being $t = 0$. Hence given C_1, a set of seven points P_7^2 is determined by seven parameters t_i $(i = 1, 2, \ldots, 7)$. If, on the other hand, seven points only are given, they determine a net of cubics containing among them twenty-four cuspidal cubics. Thus the fact that C_1 is given is equivalent to the assumption of a single solution of the cusp equation of degree 24 of the net. P_7^2 determined in this way is general.

The condition that two points coincide is

$$t_i - t_j = 0. \tag{1}$$

The condition that three points t_i be on a line is

$$\overset{3}{\Sigma} t_i = 0. \tag{2}$$

The condition that six points t_i be on a conic is

$$\overset{6}{\Sigma} t_i = 0. \tag{3}$$

The quadratic transformation A_{123} with F-points at t_1, t_2 and t_3 sends C_1 into another cuspidal cubic C_1' whose points can be named by means of the same parameter t. C_1' can be sent by a collineation into C_1. This operation sends a point t on C_1' into a point t' of C_1. To determine the effect on the parameters we note that if

$$t_i' = t_i + \tfrac{1}{3}(t_1 + t_2 + t_3), \tag{4}$$

then

$$t_i' + t_j' + t_k' = t_i + t_j + t_k + t_1 + t_2 + t_3.$$

That is, the requirement that to three points on a line correspond three points on a conic through the F-points is satisfied. This gives the effect of the transformation on an ordinary point. It is clear that the condition that a point coincide with an F-point goes into the condition that the corresponding point be on the opposite F-line. Hence

$$t'_i - t'_3 = t_i + t_1 + t_2,$$

and by means of (4) we obtain the relation

$$t'_3 = t_3 - \tfrac{2}{3}(t_1 + t_2 + t_3).$$

The effect of A_{123} is then that of the collineation on the parameters given by the equations

$$\begin{array}{ll} t'_1 = t_1 - \tfrac{2}{3}(t_1+t_2+t_3), & t'_2 = t_2 - \tfrac{2}{3}(t_1+t_2+t_3), \\ t'_3 = t_3 - \tfrac{2}{3}(t_1+t_2+t_3), & t'_i = t_i + \tfrac{1}{3}(t_1+t_2+t_3), \end{array} \quad i = 4, 5, 6, 7,$$

where t_1, t_2 and t_3 are F-points, and t_i ordinary points.

The aggregate of operations obtained by taking products of A_{123} and permutations of $t_1, t_2, \ldots, t_7$ constitute the group $T_{7,2}$ of P_7^2 on C_1. An element of $T_{7,2}$ can be looked upon as the operation of passing from one P_7^2 on C_1 to a congruent one named by seven other values t'_i. We get in this way 288 projectively distinct sets of points on C_1 congruent in some order. Hence there are $7!\,288$ projectively distinct ordered sets. $T_{7,2}$ the collineation group on the variables $t_1, \ldots, t_7$ is of order $7!\,288$.

Only the ratios of the t's are essential since the transformation $t'_i = \mu t_i$ represents a projectivity of C_1 into itself and therefore the sets $t_1, \ldots, t_7$ and $\mu t_1, \ldots, \mu t_7$ are projective.

III.

Invariants of $T_{7,2}$.

An invariant of $T_{7,2}$ is a function of the t's unaltered by the operations of $T_{7,2}$. The condition that two points coincide is sent by A_{123} into the condition that two points coincide, or that three points be on a line; the condition that three points be on a line is sent into the condition that two points coincide, that three points be on a line, or that six points be on a conic; the condition that six points be on a conic is sent into the condition that six points be on a conic or that three points be on a line. The algebraic expressions for these conditions ((1), (2) and (3) of II) are permuted as stated, but may change sign. Hence

$$I_2 = \sum_{21}(t_1 - t_2)^2 + \sum_{35}(t_1 + t_2 + t_3)^2 + \sum_{7}(t_1 + t_2 + t_3 + t_4 + t_5 + t_6)^2 = 9(3a_1^2 - 4a_2),$$

where a_i is a symmetric function of the t's of degree i, is an invariant of $T_{7,2}$ of degree 2. Likewise are found,

$$
\begin{aligned}
I_4 &= \sum_{21}(t_1-t_2)^4+\sum_{35}(t_1+t_2+t_3)^4+\sum_{7}(a_1-t_1)^4=3\,(3a_1^2-4a_2)^2,\\
I_6 &= \sum_{21}(t_1-t_2)^6+\sum_{35}(t_1+t_2+t_3)^6+\sum_{7}(a_1-t_1)^6\\
&= 27a_1^6-108a_1^4a_2+192a_1^2a_2^2-96a_2^3-72a_1^3a_3+24a_1a_2a_3-36a_3^2+72a_1^2a_4\\
&\quad +48a_2a_4-72a_1a_5-288a_6,
\end{aligned}
$$

which are invariants of degrees 4 and 6, respectively. I_4 is seen to be a multiple of I_2^2.

IV.

The Quartic C^4 Arising from a Set of Seven Points.

The plane E_x of P_7^2 is mapped upon a plane E_y by the cubic curves on P_7^2. To the cubics of E_x correspond the lines of E_y. Hence to two residual base points of a pencil of cubics of the net on P_7^2 there corresponds one point of E_y. If, however, a double point of a curve of the net on P_7^2 is taken, it alone corresponds to a point of E_y, for the two variable intersections of curves of the net have coincided. Since the Jacobian is the locus of double points of the net, the correspondence between the Jacobian of the net on P_7^2 and its map in E_y is one to one. The Jacobian of the net on P_7^2 has double points at the points of P_7^2, and being of order 6, will have $6\times 3-7\times 2=4$ variable intersections with curves of the net. That is, the map of the Jacobian squared, since pairs of points have coincided on it, is a quartic curve C^4 in E_y. The cubics of the net with a double point map into the lines of C^4, but the twenty-one degenerate cubics P_{ij}, consisting of the line t_it_j and the conic on the remaining five points, and the seven cubics P_{0i}, with a double point at a point of P_7^2, map into the twenty-eight double tangents of C^4 in such a way that the seven P_{0i} give rise to an Aronhold set. The twenty-four cuspidal cubics of the net map into the flex tangents and the twenty-four cusps into the flexes of C^4.

The operations of $T_{7,2}$ transform P_7^2 into Q_7^2, and transform the net of cubics on P_7^2 into a net on Q_7^2. The curves P_{0i} and P_{ij} of the net on P_7^2 are transformed into the curves Q_{0i} and Q_{ij} of Q_7^2. The effect of the generating transformation A_{123} on the curves P_{0i} and P_{ij} is to send them into Q_{0i} and Q_{ij} in such a way as to bring about the following permutation of the pairs of subscripts:

$$(01,23)\,(02,31)\,(03,12)\,(45,67)\,(46,57)\,(47,56),$$

the other pairs being unaltered.

Hence, the effect of the product $A_{237}A_{457}A_{167}$ is that of the interchange of subscripts 0 and 7. $G_{7!}$ together with the transposition (07) generates a subgroup of $T_{7,2}$, the symmetric group $G_{8!}$ of the permutations of 0, 1, 2, ..., 7. By comparing the notation above for the curves P with the Hesse notation of the double tangents $[ik;\ i, k=1, \ldots, 8;\ i \neq k]$ of the quartic, it is seen at once that $G_{8!}$ and A_{123} effect the same permutations on the curves P as the subgroup* E and the substitution P_{1238}, which generate the group of the double tangents, effect on the double tangents. Since the order of $T_{7,2}$ is that of the group of the double tangents of the quartic,

$T_{7,2}$ is simply isomorphic with the group of the double tangents of the quartic.

V.

A Net of Cubics on P_7^2.

We will obtain the quartic map of the Jacobian for the net of cubics on $t_1, t_2, \ldots, t_7$ formed by taking the three following base cubics

$C_1 \equiv -x_1x_3^2 + x_2^3 = 0$, the cuspidal cubic above;

$$C_2 \equiv x_1^2x_2 - a_1x_1^2x_3 + a_2x_1x_2^2 - a_3x_1x_2x_3 + a_4x_2^3 - a_5x_2^2x_3 + a_6x_2x_3^2 - a_7x_3^3 = 0,$$

the cubic on $t_1, \ldots, t_7$ having no term in $x_1x_3^2$ and passing through the cusp of C_1;

$$C_3 \equiv 4x_1^3 + x_1^2(c_2x_2 - c_3x_3) + x_1(c_4x_2^2 - c_5x_2x_3) + c_6x_2^3 - c_7x_2^2x_3 + c_8x_2x_3^2 - c_9x_3^3 = 0,$$

the cubic on $t_1, \ldots, t_7$ having no term in $x_1x_3^2$ and touching C_1 at $t = -\frac{1}{2}a_1$. The a_i are symmetric functions of $t_1, \ldots, t_7$ of degree i; the c's are given in terms of the a's by

$$c_n = 4a_n - 4a_1a_{n-1} + a_1^2a_{n-2}.$$

VI.

The Jacobian $J[C_1C_2C_3]$ and its Map C^4.

We have now to determine the Jacobian J of C_1, C_2, and C_3, and obtain the quartic map of J^2 by means of the equations

$$x_1' = C_1, \quad x_2' = C_2, \quad x_3' = C_3.$$

$$\begin{aligned} J \equiv\ & 24x_1^5x_3 - 36a_1x_1^4x_2^2 + 48a_2x_1^4x_2x_3 + (-24a_3 + 3a_1c_2 - 3c_3)x_1^4x_3^2 \\ & + (-36a_3 - 6a_1c_2 + 6c_3)x_1^3x_2^3 + (72a_4 + 6a_2c_2 - 6c_4)x_1^3x_2^2x_3 \\ & + (-48a_5 - 3a_3c_2 - 6a_2c_3 + 6a_1c_4 + 3c_5)x_1^3x_2x_3^2 + (24a_6 + 3a_3c_3 - 3a_1c_5)x_1^3x_3^3 \\ & + (-36a_5 - 6a_3c_2 + 3a_2c_3 - 3a_1c_4 + 6c_5)x_1^2x_2^4 \\ & + (72a_6 + 12a_4c_2 + 3a_3c_3 - 3a_1c_5 - 12c_6)x_1^2x_2^3x_3 \end{aligned}$$

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* "Finite Groups," Miller, Blichfeldt and Dickson, pp, 362–365.

$$
\begin{aligned}
&+(-108a_7-9a_5c_2-9a_4c_3+9a_1c_6+9c_7)x_1^2x_2^2x_3^2+(6a_6c_2+6a_5c_3-6a_1c_7-6c_8)x_1^2x_2x_3^3\\
&+(-3a_7c_2-3a_6c_3+3a_1c_8+3c_9)x_1^2x_3^4+(-6a_5c_2-3a_3c_4+3a_2c_5+6c_7)x_1x_2^5\\
&+(12a_6c_2+6a_5c_3+6a_4c_4-6a_2c_6-6a_1c_7-12c_8)x_1x_2^4x_3\\
&+(-18a_7c_2-12a_6c_3-6a_5c_4-3a_4c_5+3a_3c_6+6a_2c_7+12a_1c_8+18c_9)x_1x_2^3x_3^2\\
&+(18a_7c_3+6a_6c_4+3a_5c_5-3a_3c_7-6a_2c_8-18a_1c_9)x_1x_2^2x_3^3\\
&+(-6a_7c_4-3a_6c_5+3a_3c_8+6a_2c_9)x_1x_2x_3^4+(3a_7c_5-3a_3c_9)x_1x_3^5\\
&+(-3a_5c_4+3a_2c_7)x_2^6+(3a_6c_4+3a_5c_5-3a_3c_7-6a_2c_8)x_2^5x_3\\
&+(-9a_7c_4-6a_6c_5-3a_5c_6+3a_4c_7+6a_3c_8+9a_2c_9)x_2^4x_3^2\\
&+(9a_7c_5+6a_6c_6-6a_4c_8-9a_3c_9)x_2^3x_3^3+(-9a_7c_6-3a_6c_7+3a_5c_8+9a_4c_9)x_2^2x_3^4\\
&+(6a_7c_7-6a_5c_9)x_1x_3^5+(-3a_7c_8+3a_6c_9)x_3^6=0.
\end{aligned}
$$

$\left(\dfrac{J}{3}\right)^2$ expressed in terms of the cubics C_1, C_2, and C_3, *i. e.*, the map C^4 in E_y if the C's are regarded as reference lines is:

$$
\begin{aligned}
\left(\frac{J}{3}\right)^2 \equiv\ &(64a_1a_2a_3a_5a_7-32a_2a_3a_6a_7-64a_1a_3^2a_4a_7+32a_1a_2^2a_6a_7\\
&-64a_1^2a_2^2a_5a_7+64a_1^2a_2a_3a_4a_7-64a_4a_7^2-64a_1a_5^2a_7+128a_1^2a_4a_5a_7\\
&-64a_1^3a_4^2a_7+64a_5a_6a_7-64a_1a_4a_6a_7+16a_2^2a_7^2+16a_3^2a_6^2+16a_1^2a_2^2a_6^2\\
&-32a_1a_2a_3a_6^2)C_1^4+(8a_1^2a_3^2a_6-8a_1^3a_2a_3a_6+80a_1^3a_4a_7-72a_1^2a_2a_3a_7\\
&+16a_1^3a_2^2a_7+16a_1^2a_6^2-16a_1^3a_5a_6+16a_1^4a_4a_6-112a_1^2a_5a_7+64a_1a_3^2a_7\\
&+64a_3a_4a_7-32a_1a_2a_4a_7+64a_7^2+64a_1a_6a_7+32a_1a_3a_4a_6-32a_1^2a_2a_4a_6\\
&-32a_2a_5a_7-32a_3a_5a_6+32a_1a_2a_5a_6)C_1^3C_2+(8a_1a_2a_3a_6-8a_3^2a_6\\
&+48a_1a_4a_7+8a_2a_3a_7-16a_1a_2^2a_7-16a_6^2+16a_1a_5a_6-16a_1^2a_4a_6-16a_5a_7)C_1^3C_3\\
&+(16a_2a_6-16a_1a_2a_5+8a_1^3a_5-2a_1^2a_3^2+8a_1a_3a_4+8a_3a_5-48a_1a_7)C_1^2C_2C_3\\
&+(a_1^4a_3^2-4a_1^5a_5-8a_1^4a_6+16a_1^2a_2a_6+16a_1^3a_2a_5-8a_1^3a_3a_4-8a_1^2a_3a_5\\
&-16a_1^3a_7+32a_1a_2a_7-32a_1a_3a_6+16a_5^2-32a_1a_4a_5+16a_1^2a_4^2-64a_3a_7)C_1^2C_2^2\\
&+(a_3^2-4a_1a_5+8a_6)C_1^2C_3^2+(a_1^6-4a_1^4a_2+8a_1^3a_3-16a_1^2a_4+32a_1a_5)C_1C_2^3\\
&+(8a_1^2a_2-3a_1^4-8a_1a_3-16a_4)C_1C_2^2C_3+(3a_1^2-4a_2)C_1C_2C_3^2-C_1C_3^3+16C_2^3C_3=0.
\end{aligned}
$$

The quartic C^4 is only projectively determined since the net of cubics is only projectively determined by the choice of seven points, that is, C_1, C_2, and C_3 can be any linearly independent cubics of the net. Moreover, any transformation sending the net of cubics into a net of cubics transforms C^4 point by point into itself. Suppose such a transformation is given by the equations $x_i=q_i(x')$. Then $C_i(x)$ goes into $C_i[q_1(x')\ldots\ldots]\equiv C_i'(x')$. A point of C^4 is given by $y_i=C_i(x)$ where x is a point of J. But the transform of the point is $y_i'=C_i'(x')$. Since $C_i'(x')\equiv C_i[q_1(x')\ldots\ldots]\equiv C_i(x)$, $y_i'=y_i$ and the point is unaltered. The curves P_{0i} and P_{ij} have, however, been permuted and the coefficients are those derived from the transformed Aronhold set and are in general altered in form. For, if the transformation A_{123} is applied to the set

of points $t_1, \ldots, t_7$, and to the cubics C_1, C_2, C_3 we obtain a new set of points $t'_1, \ldots, t'_7$ and three new cubics C'_1, C'_2, C'_3 whose coefficients contain not only the symmetric functions of the points $t'_1, \ldots, t'_7$, but in addition the F-points t'_1, t'_2, and t'_3. Likewise the transform of C_4 by A_{123}, that is, the map of $J^2[C'_1, C'_2, C'_3]$ by C'_1, C'_2, and C'_3 will contain the F-points t'_1, t'_2, and t'_3, besides the symmetric functions of $t'_1, \ldots, t'_7$. If, however, C_1, C_2, and C_3 were cubics covariant under A_{123}, then since J also is covariant, the coefficients of the quartic C^4 would be invariants of A_{123}, and since they are symmetric would be invariants of $T_{7,2}$. The cubics C_1, C_2, and C_3 are mapped into the reference lines of the plane E_y. Since C_1 is covariant and is mapped into the known flex tangent, a triangle of reference determined uniquely by the flex would arise from a set of cubics covariant with the cusp cubic. The problem of finding the above-mentioned invariants of $T_{7,2}$ is then reduced to that of finding the coefficients of C^4 referred to a triangle of reference covariant with the flex.

A simple way to determine such a covariant triangle is to take

(1) for x_1 the line C_1, the flex tangent;

(2) for x_3 the tangent to C^4 at the intersection of C_1 with C^4 other than the flex $(0, 0, 1)$;

(3) for x_2 the line joining $0, 0, 1$ with the intersection of x_3 with the polar line of $0, 0, 1$ as to the polar conic of $0, 1, 0$ as to C^4.

The above choice of reference lines gives the following linear transformation of the C's to the new variables x:

$$C_1 = x_1, \quad C_2 = \lambda x_1 + x_2, \quad C_3 = \mu x_1 + x_3,$$

where $48\lambda = 3a_1^4 - 8a_1^2 a_2 + 8a_1 a_3 + 16a_4$ and $16\mu = -a_1^6 + 4a_1^4 a_2 - 8a_1^3 a_3 + 16a_1^2 a_4 - 32a_1 a_5$.

The following expressions A_i, invariants of degree i of $T_{7,2}$, are such numerical multiples of the coefficients of the transform of C^4 as to remove fractional coefficients. α_{ijkl} is the coefficient of $x_i x_j x_k x_l$ where i, j, k, l are 1, 2, or 3.

$$A_2 = \alpha_{1233} = 3a_1^2 - 4a_2.$$

$$A_6 = 48\alpha_{1133} = 18a_1^6 - 72a_1^4 a_2 + 96a_1^3 a_3 + 32a_1^2 a_2^2 - 96a_1^2 a_4 - 32a_1 a_2 a_3 + 96a_1 a_5 - 64a_2 a_4 + 48a_3^2 + 384a_6.$$

$$A_8 = 48\alpha_{1123} = -27a_1^8 + 144a_1^6 a_2 - 192a_1^5 a_3 - 160a_1^4 a_2^2 + 192a_1^4 a_4 + 320a_1^3 a_2 a_3 - 192a_1^3 a_5 - 128a_1^2 a_2 a_4 - 160a_1^2 a_3^2 + 128a_1 a_3 a_4 - 2304a_1 a_7 + 768a_2 a_6 + 384a_3 a_5 - 256a_4^2.$$

$$A_{10} = 16\alpha_{1122} = 3a_1^{10} - 20a_1^8 a_2 + 32a_1^7 a_3 + 32a_1^6 a_2^2 - 32a_1^6 a_4 - 96a_1^5 a_2 a_3 + 32a_1^5 a_5 + 64a_1^4 a_2 a_4 + 80a_1^4 a_3^2 - 128a_1^4 a_6 - 128a_1^3 a_3 a_4 - 256a_1^3 a_7 + 256a_1^2 a_2 a_6 + 128a_1^2 a_3 a_5 + 512a_1 a_2 a_7 - 512a_1 a_3 a_6 - 1024a_3 a_7 + 256a_5^2.$$

$$
\begin{aligned}
A_{12}=6912\alpha_{1113}= & -117a_1^{12}+936a_1^{10}a_2-1152a_1^9a_3-2352a_1^8a_2^2+5376a_1^7a_2a_3+4608a_1^7a_5 \\
& +1536a_1^6a_2a_4-2816a_1^6a_3^2-6912a_1^6a_6-5376a_1^5a_2^2a_3-8064a_1^5a_2a_5+1152a_1^5a_3a_4 \\
& -20736a_1^5a_7-4608a_1^4a_2^2a_4+8064a_1^4a_2a_3^2+34560a_1^4a_2a_6+23040a_1^4a_3a_5-2304a_1^4a_4^2 \\
& +12288a_1^3a_2^2a_5+2304a_1^3a_2a_3^2+3072a_1^3a_2a_3a_4+55296a_1^3a_2a_7-10240a_1^3a_3^3 \\
& -55296a_1^3a_3a_6-36864a_1^3a_4a_5-18432a_1^2a_2^2a_6-21504a_1^2a_2a_3a_5+6144a_1^2a_2a_4^2 \\
& +12288a_1^2a_3^2a_4-55296a_1^2a_3a_7+101376a_1^2a_5^2-110592a_1a_2^2a_7+73728a_1a_2a_3a_6 \\
& -24576a_1a_2a_4a_5-18432a_1a_3^2a_5+6144a_1a_3a_4^2+22184a_1a_4a_7-110592a_1a_5a_6 \\
& +55296a_2a_3a_7+36864a_2a_4a_6-55296a_3^2a_6+13824a_3a_4a_5-8192a_4^3-110592a_5a_7 \\
& -110592a_6^2.
\end{aligned}
$$

$$
\begin{aligned}
A_{14}=768\alpha_{1112}= & \,27a_1^{14}-252a_1^{12}a_2+384a_1^{11}a_3+752a_1^{10}a_2^2-384a_1^{10}a_4-2176a_1^9a_2a_3 \\
& +384a_1^9a_5-608a_1^8a_2^3+1792a_1^8a_2a_4+1568a_1^8a_3^2-768a_1^8a_6+2816a_1^7a_2^2a_3 \\
& -1280a_1^7a_2a_5-2944a_1^7a_3a_4+768a_1^7a_7-1536a_1^6a_2^2a_4-3968a_1^6a_2a_3^2+2816a_1^6a_2a_6 \\
& +2432a_1^6a_3a_5+1280a_1^6a_4^2+4608a_1^5a_2a_3a_4-2048a_1^5a_2a_7+2048a_1^5a_3^3-5120a_1^5a_3a_6 \\
& -2048a_1^5a_4a_5-1024a_1^4a_2^2a_6-512a_1^4a_2a_3a_5-1024a_1^4a_2a_4^2-4096a_1^4a_3^2a_4 \\
& +8192a_1^4a_3a_7+8192a_1^4a_4a_6-1536a_1^4a_5^2+4096a_1^3a_2^2a_7+2048a_1^3a_3^2a_5+2048a_1^3a_3a_4^2 \\
& +16384a_1^3a_4a_7-12288a_1^3a_5a_6-30720a_1^2a_2a_3a_7-4096a_1^2a_2a_4a_6+8192a_1^2a_2a_5^2 \\
& -2048a_1^2a_3^2a_6-2048a_1^2a_3a_4a_5-12288a_1^2a_5a_7+12288a_1^2a_6^2-8192a_1a_2a_4a_7 \\
& +32768a_1a_3^2a_7+8192a_1a_3a_4a_6-8192a_1a_3a_5^2+49152a_1a_6a_7-24576a_2a_5a_7 \\
& +16384a_3a_4a_7-24576a_3a_5a_6+8192a_4a_5^2+49152a_7^2.
\end{aligned}
$$

$$
\begin{aligned}
A_{18}=9\cdot\overline{16}^3\alpha_{1111}= & \,63a_1^{18}-756a_1^{16}a_2+1152a_1^{15}a_3+3264a_1^{14}a_2^2-1152a_1^{14}a_4-9600a_1^{13}a_2a_3 \\
& +1728a_1^{13}a_5-5600a_1^{12}a_2^3+8448a_1^{12}a_2a_4+7056a_1^{12}a_3^2-1152a_1^{12}a_6+24576a_1^{11}a_2^2a_3 \\
& -12288a_1^{11}a_2a_5-13824a_1^{11}a_3a_4+4608a_1^{11}a_7+2816a_1^{10}a_2^4-16896a_1^{10}a_2^2a_4 \\
& -34944a_1^{10}a_2a_3^2+6144a_1^{10}a_2a_6+21504a_1^{10}a_3a_5+6912a_1^{10}a_4^2-17152a_1^9a_2^3a_3 \\
& +22016a_1^9a_2^2a_5+53760a_1^9a_2a_3a_4-29184a_1^9a_2a_7+17408a_1^9a_3^3-10752a_1^9a_3a_6 \\
& -30720a_1^9a_4a_5+6656a_1^8a_2^3a_4+34816a_1^8a_2^2a_3^2-5120a_1^8a_2^2a_6-76288a_1^8a_2a_3a_5 \\
& -21504a_1^8a_2a_4^2-46080a_1^8a_3^2a_4+52224a_1^8a_3a_7+61440a_1^8a_4a_6+20736a_1^8a_5^2 \\
& -20480a_1^7a_2^2a_3a_4+106496a_1^7a_2^2a_7-31744a_1^7a_2a_3^3-8192a_1^7a_2a_3a_6+114688a_1^7a_2a_4a_5 \\
& +72704a_1^7a_3^2a_5+36864a_1^7a_3a_4^2-24576a_1^7a_4a_7-73728a_1^7a_5a_6-8192a_1^6a_2^3a_6 \\
& -4096a_1^6a_2^2a_3a_5-8192a_1^6a_2^2a_4^2+30720a_1^6a_2a_3^2a_4-299008a_1^6a_2a_3a_7 \\
& -237568a_1^6a_2a_4a_6-61440a_1^6a_2a_5^2+11264a_1^6a_3^4+4096a_1^6a_3^2a_6-225280a_1^6a_3a_4a_5 \\
& -8192a_1^6a_4^3+73728a_1^6a_6^2-212992a_1^5a_3^3a_7+229376a_1^5a_3^2a_7-4096a_1^5a_2a_3^2a_5 \\
& +24576a_1^5a_2a_3a_4^2+139264a_1^5a_2a_4a_7+245760a_1^5a_2a_5a_6-12288a_1^5a_3^3a_4 \\
& +229376a_1^5a_3^2a_7+385024a_1^5a_3a_4a_6+172032a_1^5a_3a_5^2+180224a_1^5a_4^2a_5 \\
& +147456a_1^5a_6a_7+778240a_1^4a_2^2a_3a_7+81920a_1^4a_2^2a_4a_6-81920a_1^4a_2^2a_5^2 \\
& -286720a_1^4a_2a_3^2a_6+40960a_1^4a_2a_3a_4a_5-32768a_1^4a_2a_4^3-122880a_1^4a_2a_5a_7 \\
& -245760a_1^4a_2a_6^2+8192a_1^4a_3^3a_5-36864a_1^4a_3^2a_4^2-409600a_1^4a_3a_4a_7-466944a_1^4a_3a_5a_6 \\
& -425984a_1^4a_4^2a_6-466944a_1^4a_4a_5^2+147456a_1^4a_7^2-327680a_1^3a_2^2a_4a_7-819200a_1^3a_2a_3^2a_7 \\
& +163840a_1^3a_2a_3a_5^2-393216a_1^3a_2a_6a_7+163840a_1^3a_3^3a_6-16384a_1^3a_3^2a_4a_5
\end{aligned}
$$

$$+65536a_1^3a_3a_4^3+196608a_1^3a_3a_5a_7+393216a_1^3a_3a_6^2-262144a_1^3a_4^2a_7$$
$$+1572864a_1^3a_4a_5a_6+294912a_1^3a_5^3-983040a_1^2a_2^2a_5a_7+589824a_1^2a_2^2a_6^2$$
$$+1572864a_1^2a_2a_3a_4a_7-393216a_1^2a_2a_3a_5a_6-131072a_1^2a_2a_4^2a_6+131072a_1^2a_2a_4a_5^2$$
$$-393216a_1^2a_2a_7^2+327680a_1^2a_3^3a_7-131072a_1^2a_3^2a_4a_6-81920a_1^2a_3^2a_5^2$$
$$-65536a_1^2a_3a_4^2a_5+393216a_1^2a_3a_6a_7+393216a_1^2a_4a_5a_7-393216a_1^2a_4a_6^2$$
$$-1179648a_1^2a_5^2a_6+1179648a_1a_2^2a_6a_7-1179648a_1a_2a_3a_6^2-262144a_1a_2a_4^2a_7$$
$$+393216a_1a_3^2a_5a_6+262144a_1a_3a_4^2a_6-131072a_1a_3a_4a_5^2+393216a_1a_3a_7^2$$
$$-1572864a_1a_4a_6a_7-1179648a_1a_5^2a_7+1179648a_1a_5a_6^2+589824a_2^2a_7^2$$
$$-1179648a_2a_3a_6a_7-393216a_2a_4a_5a_7+589824a_3^2a_6^2+524288a_3a_4^2a_7$$
$$-393216a_3a_4a_5a_6+65536a_4^2a_5^2-1572864a_4a_7^2+2359296a_5a_6a_7.$$

The equation of C^4 referred to the chosen reference lines covariant with the flex is

$$3A_{18}x_1^4+144A_{14}x_1^3x_2+16A_{12}x_1^3x_3+6912A_{10}x_1^2x_2^2+2304A_8x_1^2x_2x_3$$
$$+2304A_6x_1^2x_3^2+110592A_2x_1x_2x_3^2-110592x_1x_3^3+1769472x_2^3x_3=0.$$

VII.

Double Tangents of C^4.

The double tangents of C^4 in E_y are of two types:

(1) The type $(0i)$ are the maps of the cubic curves P_{0i} of the net in E_x with a double point at t_i. There are seven of this type and they form an Aronhold set.

(2) The type (ij) are the maps of the cubic curves P_{ij} of the net in E_x consisting of the line t_it_j and the conic on the remaining five points. There are twenty-one of this type.

To determine the equation of the double tangent $(0i)$ the equation of P_{0i} must first be found. The curves of the net having a common tangent at t_i form a pencil whose equation is

$$k(x_2^3-x_1x_3^2)+x_1^3+(a_2-a_1^2-a_1t_i-t_i^2)x_1^2x_2-(a_3-a_1a_2-a_1^2t_i-a_1t_i^2)x_1^2x_3$$
$$+(a_4-a_1a_3-a_1a_2t_i-a_2t_i^2)x_1x_2^2-(a_5-a_1a_4-a_1a_3t_i-a_3t_i^2)x_1x_2x_3$$
$$+(a_6-a_1a_5-a_1a_4t_i-a_4t_i^2)x_1x_3^2-(a_7-a_1a_6-a_1a_5t_i-a_5t_i^2)x_2^2x_3$$
$$+(-a_1a_7-a_1a_6t_i-a_6t_i^2)x_2x_3^2-(-a_1a_7t_i-a_7t_i^2)x_3^3=0.$$

There is a single member of this pencil with a double point at t_i. This is the curve for which k has such a value that the point t_i, when substituted in a derivative with respect to x_1 makes it vanish. This value of k is

$$k=t_i^6+a_2t_i^4+(a_1a_2-a_3)t_i^3-a_5t_i+(a_6-a_1a_5).$$

The map in E_y of the curve of the above pencil for this value of k, that is, the equation of the double tangent $(0i)$ is

$$4[t_i^6+a_2t_i^4+(a_1a_2-a_3)t_i^3-a_5t_i+(a_6-a_1a_5)]C_1-(a_1+2t_1)^2C_2+C_3=0.$$

The equation of the line t_it_j is

$$x_1+(s_2-s_1^2)x_2+s_1s_2x_3=0,$$

where s_k is the symmetric function of t_i and t_j of degree k. The equation of the conic on the remaining five points is

$$x_1^2+(\sigma_4-\sigma_1\sigma_3)x_2^2-\sigma_1\sigma_5x_3^2+(\sigma_2-\sigma_1^2)x_1x_2-(\sigma_3-\sigma_1\sigma_2)x_1x_3-(\sigma_5-\sigma_1\sigma_4)x_2x_3=0,$$

where σ_k is the symmetric function of the five points other than t_i and t_j. To obtain the equation of the double tangent (ij) in E_y we have to find the map of the product of the equations of the line and conic above. This product expressed in terms of C_1, C_2, and C_3, that is, the equation of the double tangent (ij) is, after removing numerical fractions

$$4(\sigma_1\sigma_5+s_1s_2\sigma_3-s_1s_2\sigma_1\sigma_2)C_1-(a_1-2s_1)^2C_2+C_3=0.$$

VIII.

Proof of the Completeness of the System of Invariants.

The determination of the t's depends on the separation of the double tangents of the quartic and the isolation of a single flex. The t's are then projective irrational invariants of the quartic. Any function of the t's of proper weight is therefore a projective invariant of the quartic. Hence any invariant of $T_{7,2}$ is an irrational invariant of the quartic, such that the only irrationality present is that of the flex. Such an invariant is expressible rationally and integrally in terms of the coefficients of the quartic and of the coordinates of the isolated flex. But since the quartic has for coefficients the invariants A_2, A_6, A_8, A_{10}, A_{12}, A_{14}, and A_{18}, and the coordinates of the flex are $0, 0, 1$, every invariant of $T_{7,2}$ is rationally and integrally expressible in terms of the invariants A_2, A_6, A_8, A_{10}, A_{12}, A_{14}, and A_{18}. Moreover, it is obvious when special values are given to $a_1, a_2, \ldots, a_7$ that no one of the invariants A_i can be expressed rationally and integrally in terms of the others. Hence none of them is superfluous and

The invariants A_2, A_6, A_8, A_{10}, A_{12}, A_{14}, and A_{18} form a complete system for the group $T_{7,2}$.

IX.

The Jacobian of $A_2, A_4, \ldots, A_{18}$.

If an expression is alternating under operations of $T_{7,2}$ it contains as a factor $t_i - t_j$ and all its conjugate values under the operations of $T_{7,2}$. Hence it has as a factor

$$J \equiv \prod^{21}(t_i - t_j) \prod^{35}(t_i + t_j + t_k) \prod^{7}(a_1 - t_i),$$

which is an alternating expression of degree 63.

The Jacobian of $A_2, A_4, \ldots, A_{18}$ is an alternating expression of degree 63 and is therefore to within a numerical factor the product J. Since J at most changes sign when the operations of $T_{7,2}$ are carried out, its square is an invariant of degree 126 and is rationally expressible in terms of the invariants $A_2, A_4, \ldots, A_{18}$.

X.

The Group $T_{8,3}$ of P_8^3 a Set of Base-Points of a Net of Quadrics.

There are in general an infinite number of projectively distinct sets of eight points in space congruent to a single set P_8^3 under a Cremona transformation which can be decomposed into a product of cubic Cremona transformations with F-points at points of P_8^3. If, however, P_8^3 is a set of base-points of a net of quadrics we can make use of the following theorem:

*If P_8^3 is a set of base-points of a net of quadrics, there are only thirty-six projectively distinct sets congruent in some order to P_8^3.**

There are then thirty-six types of congruence if no account is taken of the order of the points. If we require that P_8^3 be an ordered set we have, since P_8^3 can be ordered in $8!$ ways, $8!36$ types of congruence. The aggregate of operations transforming P_8^3 into the $8!36$ congruent sets constitute a group which we will call $T_{8,3}$. Any one of these operations is, as presupposed, the product of cubic transformations which can be obtained from a single one by a permutation of the points of P_8^3. Hence $T_{8,3}$ is generated by a cubic transformation and the symmetric group of permutations of the points of P_8^3 of order $8!$. Abstractly then $T_{8,3}$ has as generators precisely the set to which the generators of $T_{7,2}$ were shown to be equivalent in IV. $T_{8,3}$ and $T_{7,2}$ are therefore abstractly the same group.

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* Coble, "Point Sets and Allied Cremona Groups," Part II, *Trans. Am. Math. Soc.*, Vol. XVII, p. 377 (45).

XI.

P_8^3 on a Cuspidal Quartic.

A cuspidal quartic curve D in space is determined by the parametric equations

$$x_1=t^4,\quad x_2=t^2,\quad x_3=t,\quad x_4=1.$$

The condition that two points t_i and t_j coincide is $t_i-t_j=0$.

The condition that four points be on a plane is $\overset{4}{\Sigma}t_i=0$.

The condition that eight points be on a quadric is $\overset{8}{\Sigma}t_i=0$.

XII.

A Net of Quadrics on P_8^3. Generators of $T_{8,3}$.

Since we wish to consider P_8^3 as the set of base-points of a net of quadrics, we determine P_8^3 by a choice of eight values t_i subject to the single condition $\overset{8}{\Sigma}t_i=0$, for the quartic above is the intersection of the quadrics

$$Q_2\equiv x_1x_4-x_2^2=0 \quad\text{and}\quad Q_3\equiv x_2x_4-x_3^2=0.$$

A third quadric on P_8^3 is

$$Q_1\equiv x_1^2+b_2x_1x_2-b_3x_1x_3+b_4x_2^2-b_5x_2x_3+b_6x_3^2-b_7x_3x_4+b_8x_4^2=0,$$

where b_i is the symmetric function of the t's of degree i. Hence:

P_8^3 determined by the choice of eight t's subject to the single condition $b_1=0$ is the set of base-points of the net of quadrics

$$y_1Q_1+y_2Q_2+y_3Q_3=0.$$

Generators of the group $T_{8,3}$ of P_8^3 determined in this way consist of the symmetric group of permutations of the eight t's together with a transformation on the t's corresponding to a cubic Cremona transformation A_{1234} with F-points at points of P_8^3, say at t_1, t_2, t_3, t_4; for the effect of A_{1234} is to send the net of quadrics into a net of quadrics, and the cuspidal quartic D into another cuspidal quartic D' whose points are named by the same parameter t. The quartic D' can be sent back into D by means of a collineation carrying the point t of D' into the point t' of D. Thus, can the transformation A_{1234} be regarded as a transformation upon the parameters t to new parameters t'.

The transformation

$$\begin{aligned} t_i'&=t_i-\tfrac{1}{2}(t_1+t_2+t_3+t_4) \qquad (i=1,2,3,4),\\ t_j'&=t_j+\tfrac{1}{2}(t_1+t_2+t_3+t_4) \qquad (j=5,6,7,8),\end{aligned}$$

is seen to have the effect of permuting the conditions that two points coincide, that four points be on a plane as does the transformation A_{1234}, and thus gives the effect of A_{1234} upon P_8^3 in terms of the parameters t_i.

XIII.

The Quartic D^4.

If we consider y_1, y_2, y_3 as the coordinates of a point in a plane, we have by means of the net of quadrics

$$y_1Q_1+y_2Q_2+y_3Q_3=0,$$

a correspondence between the points y of a plane and the quadrics (yQ) of the net. To a pencil of quadrics or the elliptic quartic curve carrying the pencil corresponds a line of the plane. Corresponding to the quadrics of the net with a double point we will have a certain locus of points in the plane. Since in each pencil of the net are four quadrics with a double point, the locus is a quartic curve. Its equation in variables y_i found by writing the discriminant of the net is

$$D^4 \equiv \begin{vmatrix} 2y_1 & b_2y_1 & -b_3y_1 & y_2 \\ b_2y_1 & 2(b_4y_1-y_2) & -b_5y_1 & y_3 \\ -b_3y_1 & -b_5y_1 & 2(b_6y_1-y_3) & -b_7y_1 \\ y_2 & y_3 & -b_7y_1 & 2b_8y_1 \end{vmatrix}$$

$$\begin{aligned} &\equiv(-4b_2^2b_6b_8+b_2^2b_7^2+4b_2b_3b_5b_8-4b_3^2b_4b_8+16b_4b_6b_8-4b_4b_7^2)\,y_1^4 \\ &\quad+(-2b_2b_5b_7+4b_3^2b_8+4b_3b_4b_7-16b_6b_8+4b_7^2)\,y_1^3y_2 \\ &\quad+(4b_2^2b_8-2b_2b_3b_7-16b_4b_8+4b_5b_7)\,y_1^3y_3+(-4b_3b_7-4b_4b_6+b_5^2)\,y_1^2y_2^2 \\ &\quad+(4b_2b_6-2b_3b_5+16b_8)\,y_1^2y_2y_3+(b_3^2-4b_6)\,y_1^2y_3^2+4b_6y_1y_2^3 \\ &\quad+4b_4y_1y_2^2y_3-4b_2y_1y_2y_3^2+4y_1y_3^3-4y_2^3y_3=0. \end{aligned}$$

XIV.

Complete System for $T_{8,3}$.

The quartic D^4 consists of precisely the same terms as does C^4 above. The flex tangent now is the map of the quartic D which is unaltered by the operations of $T_{8,3}$. If we choose as base quadrics of the net quadrics covariant with D, since the discriminant is an invariant of the net, its coefficients will be invariants of $T_{8,3}$. Since the quartic D maps into the flex tangent, we need not determine these quadrics, but have only to choose in the plane a triangle of reference covariant with the flex. The lines are chosen in the same way as for C^4. The transformation is therefore of the same type as (5), removes the same terms from D^4 as (5) does from C^4 and is

$$y_1=3y_1', \quad y_2=b_4y_1'+3y_2', \quad y_3=3b_6y_1'+3y_3'.$$

The transformed expression for D^4 is

$$3B_{18}y_1'^4+27B_{14}y_1'^3y_2'+3B_{12}y_1'^3y_3'+81B_{10}y_1'^2y_2'^2+27B_8y_1'^2y_2'y_3'$$
$$+27B_6y_1'^2y_3'^2+81B_2y_1'y_2'y_3'^2+324y_1'y_3'^3-324y_2'^3y_3'=0,$$

where B_i is an invariant of $T_{8,3}$ of degree i, whose explicit expressions are as follows:

$$\begin{aligned}
B_2 &=-4b_2,\\
B_6 &=-4b_2b_4+3b_3^2+24b_6,\\
B_8 &=-12b_2b_6-6b_3b_5+4b_4^2+48b_8,\\
B_{10} &=-4b_3b_7+b_5^2,\\
B_{12} &=54b_3^2b_6-18b_3b_4b_5+8b_4^3+108b_6^2,\\
B_{14} &=-6b_2b_5b_7-6b_3b_5b_6+12b_3^2b_8+4b_3b_4b_7+2b_4b_5^2+12b_7^2,\\
B_{18} &=27b_2^2b_7^2+108b_2b_3b_5b_8-54b_2b_3b_6b_7-18b_2b_4b_5b_7-72b_3^2b_4b_8+27b_3^2b_6^2\\
&\quad+24b_3b_4^2b_7-18b_3b_4b_5b_6+3b_4^2b_5^2-72b_4b_7^2-108b_5^2b_8+108b_5b_6b_7.
\end{aligned}$$

It is to be noted that great simplicity is gained in the complete system when the group is represented in eight variables whose sum is zero. This is to be expected since every term in which b_1 enters vanishes. Moreover, the notation for the double tangents is symmetrical.

The lines of D^4 arise from the pencils of quadrics of the net. The quadrics in a pencil with a double point correspond to the meets of the line with D^4. To a pencil of quadrics such that the double points have coincided in pairs corresponds a double tangent of D^4. Such a pencil of the net can be found by requiring that it contain the line t_it_j. To do this we have only to require that the quadric contain a point of the line t_it_j other than t_i and t_j. That is, the linear condition on the y''s is

$$(3\sigma_6+3s_2\sigma_4+3s_1s_2\sigma_3+3s_1^2s_2\sigma_2-s_1^2b_4-3b_6)y_1'-3s_1^2y_2'-3y_3'=0,$$

where s_k is the symmetric function of t_i and t_j of degree k, and σ_k the symmetric function of the remaining six t's. This is the condition in the plane that the point y be on a double tangent, that is, it is the equation of the double tangent corresponding to a choice of two of the eight points of P_8^3. The twenty-eight double tangents are thus all accounted for and are all of one type (ij).